

Mass angular momentum and charge inequalities for black holes in Einstein-Maxwell-axion-dilaton gravity

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Mass angular momentum and charge inequalities for axisymmetric maximal time-symmetric initial data invariant under an action of $U(1)$ group, in Einstein-Maxwell-axion-dilaton gravity being the low-energy limit of the heterotic string theory, is established. We assume that data set with two asymptotically flat regions is given on smooth simply connected manifold. We also pay attention to the area momentum charge inequalities for a closed orientable two-dimensional spacelike surface embedded in the spacetime of the considered theory.

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I. INTRODUCTION

Description of the gravitational collapse dynamics is a real challenge to theoretical investigations in the realm of Einstein gravity and its generalizations. Recently, one can observe a big resurgence of the aforementioned problems originated from the researches conducted in Ref.[1]. Revisiting original Brill's proof of positive mass [2], Riemannian Penrose inequality was proved in four and generalized to five-dimensional case of Einstein theory of gravity, in the context of time-symmetric, axisymmetric initial data [3]. The total mass angular momentum inequality, binding global quantities has been expanded to the dynamical case of vacuum and electrovacuum axisymmetric spacetimes [4]-[9]. Further perspicacity into the investigations in question was taking into considerations quasi-local quantities characterizing black holes. The aforementioned inequalities were studied in axisymmetric spacetime with matter besieged the event horizon [10]. The vacuum Einstein gravity case was treated in Refs. [11]-[12], where an inequality between area of the apparent horizon and angular momentum for a class of axially symmetric black holes including initial conditions with isometry leaving fixed two-surfaces was conceived. The initial data set of Einstein vacuum equations with cosmological constant was treated in [13]. On the other hand, the extension in order to incorporate electric and magnetic charges into the problem in question was elaborate in [14]-[16]. See also [17] and references therein, for the recent review of the problem and main ideas standing behind the proofs. The lower bound for a single black hole in Einstein-Maxwell theory with axially symmetric maximal initial data and non-electromagnetic matter fields satisfying dominant energy conditions was found in Ref.[19] (see also [18]). This inequality is saturated only for the case when the initial data arise from extreme Kerr-Newmann spacetime.

A natural extension of the predicament in question is related to the problem of gravitational collapse in generalization of Einstein theory to higher dimensions and emergence of higher dimensional black objects. The complete classification of n -dimensional charged black holes both with non-degenerate and degenerate component of the event horizon was proposed in Refs.[20] but there were only partial results for the highly nontrivial case of n -dimensional rotating black hole uniqueness theorem [21]. These researches encompasses also the case of the low-energy limit of the string theory, like dilaton gravity, Einstein-Maxwell-axion-dilaton (EMAD)-gravity and supergravities theories [22]. On the other hand, the strictly stationary static vacuum spacetimes in Einstein-Gauss-Bonnet theory were discussed in [23], while in Ref.[24] it was revealed that strictly stationary AdS spacetime could not allow for the existence of nontrivial configurations of complex scalar fields or form fields. The generalization of the aforementioned problem, i.e., strict stationarity of spacetimes with complex scalar fields in EMAD-gravity with negative cosmological constant was given in [25]. In Ref. [24] it was revealed that a static asymptotically flat black hole solution is unique to be Schwarzschild spacetime in Chern-Simons modified gravity. Then, the uniqueness proof of static asymptotically flat electrically charged black hole in Chern-Simons modified gravity was provided [26].

Just, the inequalities between area and angular momentum in higher dimensional axisymmetric spacetime were given in [27], whereas inequalities binding area angular momentum and charges in Einstein-dilaton gravity were proposed in Ref.[28]. The five-dimensional extension of the dilaton gravity was elaborated in [29]. One should also mention [30], where the inequalities for stable marginally outer trapped surfaces in dilaton gravity were derived.

Motivated by the aforementioned researches we shall search for the lower bound for the area of black holes in EMAD-gravity being the low-energy limit of heterotic string compactified to four-dimensions. One will not restrict

himself to only one gauge field and take into account the arbitrary number of $U(1)$ -gauge fields.

Our paper is organized as follows. In Sec.II we present the underlying theory, defining the $SL(2, R)$ -duals to the gauge fields and complex scalar *axi-dilaton*. Next, in Sec.III we find the general form of the total angular momentum and twist potential in the EMAD-gravity. We find inequality binding angular momentum and *dilaton-electric* and *dilaton-magnetic* charges for a black hole with axially symmetric maximal initial data as well as non-electromagnetic fields fulfilling the dominant energy condition. Sec.IV will be devoted to the area angular momentum charge inequalities for a closed orientable two-dimensional spacelike surface in manifold under consideration.

II. EMAD-GRAVITY

Motivated by the recent works connected with inequalities binding black hole mass and other its parameters in Einstein-Maxwell (EM) theory [19], we shall pose a question about such kind of inequalities in generalized theory of gravity. Namely, in this section, we consider the so-called Einstein-Maxwell-axion-dilaton gravity (EMAD). The theory under consideration will contain gravitation field $g_{\mu\nu}$, arbitrary number N of $U(1)$ -gauge fields, the dilaton field ϕ and axion a . The action for EMAD-gravity will be subject to the relation [31]

$$S = \int d^4x \frac{\sqrt{-g}}{16\pi} \left[R - 2 \nabla_\mu \phi \nabla^\mu \phi - \frac{1}{2} \nabla_\mu a \nabla^\mu a - \sum_{n=1}^N e^{-2\phi} F_{\mu\nu}^{(n)} F^{\mu\nu (n)} - \sum_{n=1}^N a F_{\mu\nu}^{(n)} * F^{\mu\nu (n)} \right], \quad (1)$$

where we have denoted the strength of the adequate gauge field $F_{\mu\nu}^{(n)} = 2 \nabla_{[\mu} A_{\nu]}^{(n)}$. On the other hand, its dual is given by $* F_{\mu\nu}^{(n)} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta (n)}$. It should be remarked that when the number of vector fields is six we obtain $N = 4$, $d = 4$ bosonic part of supergravity theory. In what follows, for the sake of generality, one will keep the arbitrary number of $U(1)$ -gauge fields.

It turned out that in many physical problems [31] the action describing by the relation (1) can be written in a more convenient form. Namely, introducing a complex scalar *axi-dilaton* in the form as

$$\lambda = a + i e^{-2\phi}, \quad (2)$$

and defining $SL(2, R)$ -duals to the gauge fields $F_{\mu\nu}^{(n)}$, the action in question implies

$$S = \int d^4x \frac{\sqrt{-g}}{16\pi} \left[R + 2 \frac{\nabla_\mu \lambda \nabla^\mu \bar{\lambda}}{(\lambda - \bar{\lambda})^2} + \sum_{n=1}^N F_{\mu\nu}^{(n)} * \tilde{\mathcal{F}}^{\mu\nu (n)} \right], \quad (3)$$

where the $SL(2, R)$ -duals are given by the relation

$$\tilde{\mathcal{F}}_{\alpha\beta}^{(n)} = e^{-2\phi} * F_{\alpha\beta}^{(n)} - a F_{\alpha\beta}^{(n)}. \quad (4)$$

The equation of motion for $SL(2, R)$ -duals is of the form $\nabla_\alpha * \tilde{\mathcal{F}}^{\alpha\beta (n)} = 0$ and entails the existence of N vector potentials $\tilde{\mathcal{A}}_\beta^{(n)}$ satisfying relation

$$\tilde{\mathcal{F}}_{\alpha\beta}^{(n)} = 2 \nabla_{[\alpha} \tilde{\mathcal{A}}_{\beta]}^{(n)}. \quad (5)$$

Consequently, the analogous relation for $F_{\mu\nu}^{(n)} = 2 \nabla_{[\mu} A_{\nu]}^{(n)}$ is not a consequence of equations of motion but it stems from the Bianchi identity. The energy momentum tensor for the complex scalar field and $U(1)$ -gauge fields is provided by the following expression:

$$T_{\alpha\beta}(F, \tilde{F}, \lambda) = - \left[4 \sum_{n=1}^N F_{\alpha\delta}^{(n)} * \tilde{\mathcal{F}}_\beta^{\delta (n)} - g_{\alpha\beta} \sum_{n=1}^N F_{\alpha\delta}^{(n)} * \tilde{\mathcal{F}}^{\alpha\delta (n)} \right] + \frac{2 g_{\alpha\beta} \nabla_\gamma \lambda \nabla^\gamma \bar{\lambda} - 4 \nabla_\alpha \lambda \nabla_\beta \bar{\lambda}}{(\lambda - \bar{\lambda})^2}. \quad (6)$$

III. MASS INEQUALITIES FOR BLACK HOLE IN EMAD-GRAVITY

First we comment on the initial value formulation of EMAD-gravity equations with matter sources. One assumes further that we have to do with non-electromagnetic matter fields. We foliate the globally hyperbolic spacetime by

Cauchy surfaces Σ_t , which are parameterized by a global time t . Let n_α be the unit normal to the aforementioned hypersurface, then $n_\alpha n^\alpha = -1$. Just the spacetime metric induced on Σ_t a spatial metric $h_{\alpha\beta}$, by the relation

$$h_{\alpha\beta} = g_{\alpha\beta} + n_\alpha n_\beta. \quad (7)$$

On this account, we define *electric* and *magnetic* components for gauge field strengths $F_{\alpha\beta}^{(n)}$ and $\tilde{F}_{\alpha\beta}^{(n)}$. Namely, *electric* components imply

$$E^{\alpha(n)} = -F^{\beta\alpha(n)} n_\beta, \quad \tilde{E}^{\alpha(n)} = -\tilde{F}_{\beta\alpha(n)} n_\beta, \quad (8)$$

while *magnetic* ones are provided by the following relations:

$$B^{\alpha(n)} = - * F^{\gamma\alpha(n)} n_\gamma, \quad \tilde{B}^{\alpha(n)} = - * \tilde{F}^{\gamma\alpha(n)} n_\gamma, \quad (9)$$

where $*$ denotes, respectively

$$* F^{\gamma\delta(n)} = \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} k^\beta F^{\gamma\delta(n)}, \quad (10)$$

$$* \tilde{F}^{\gamma\delta(n)} = \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} k^\beta \tilde{F}^{\gamma\delta(n)}. \quad (11)$$

A complete initial data for the theory provided by the action (3) will consist of the initial Cauchy hypersurface Σ_t , induced metric on it, its extrinsic curvature K_{ij} , the value of *axi-dilaton* complex scalar λ and $n^j D_j \lambda$ on Σ_t and *electric* and *magnetic* fields for each of the n -th gauge components defined on the hypersurface in question. Moreover, if Σ_t is time symmetric one has that $n^j D_j \lambda = 0$.

By virtue of the above definitions and properties of the *electric* and *magnetic* components of the adequate gauge strength field $F_{\alpha\beta}^{(n)}$ and $\tilde{F}_{\alpha\beta}^{(n)}$, one concludes that the constraint equations for EMAD-gravity are provided by

$$D^a \left(K_{ab} - K_c^c h_{ab} \right) - 2 \sum_{n=1}^N \epsilon_{bdj} B^{(n)j} \tilde{B}^{(n)d} = 8\pi P_b, \quad (12)$$

$${}^{(3)}R + (K_a^a)^2 - K_{ij} K^{ij} - 2 \left[\sum_{n=1}^N \left(B_i^{(n)} \tilde{E}^{(n)i} - E_j^{(n)} \tilde{B}^{(n)j} \right) - \frac{\chi_a \tilde{\chi}^a}{(\lambda - \bar{\lambda})^2} \right] = 16\pi \mu, \quad (13)$$

where $\chi_a = D_a \lambda$, D_a is the derivate with respect to h_{ab} metric while P_b matter momentum density and μ is matter energy density. In our considerations we assume that matter fields will satisfy the dominant energy condition $\mu \geq |P_i|$. Thus, equations (12) and (13) define the time symmetric initial data for the theory under consideration.

In what follows we shall consider asymptotically flat Riemannian manifold in which there exists a region diffeomorphic to $R^3 \setminus B(R)$, where $B(R)$ is a coordinate ball of radius R . In local coordinates on the above region the adequate fall-off conditions are required to satisfy

$$h_{ij} - \delta_{ij} = \mathcal{O}_k(r^{-\frac{1}{2}}), \quad \partial_k h_{ij} \in L^2(M_{ext}), \quad K_{ij} = \mathcal{O}_{l-1}(r^{-3}), \quad (14)$$

$$\begin{aligned} E^i &= \mathcal{O}_{l-1}(r^{-2}), & \tilde{E}^i &= \mathcal{O}_{l-1}(r^{-2}), \\ B^i &= \mathcal{O}_{l-1}(r^{-2}), & \tilde{B}^i &= \mathcal{O}_{l-1}(r^{-2}), \end{aligned} \quad (15)$$

where we have denoted $f = \mathcal{O}_k(r^\lambda)$, $\partial_{k_1 \dots k_l} f = \mathcal{O}(r^{\lambda-l})$, for $0 \leq l \leq k$.

Next, one commences with the initial data set for EMAD-gravity consisting of metric tensor, extrinsic curvature and vector connected with the gauge fields in the underlying theory, i.e., $(M, h_{ij}, K_{ab}, E_i, \tilde{E}_i, B_a, \tilde{B}_a, \lambda)$. Furthermore, in Ref.[6] it was revealed that in the case of simply connectedness of the manifold in question, the analysis reduced to

the considerations of manifold $R^3 \setminus \sum_{j=1}^H a_j$, where a_j are points in R^3 representing asymptotic ends. There also exists a global cylindrical Brill coordinate system, where a_j lie on z -axis. The fall-off conditions in asymptotically flat ends avouch the definitions of the ADM mass and the adequate charges

$$m = \frac{1}{16\pi} \int_{S_{r \rightarrow \infty}} dS \left(h_{ij,i} - h_{ii,j} \right) \tilde{n}^j, \quad (16)$$

$$Q_e^{(n)} = \frac{1}{4\pi} \int_{S_{r \rightarrow \infty}} dS E_a \tilde{n}^a, \quad Q_m^{(n)} = \frac{1}{4\pi} \int_{S_{r \rightarrow \infty}} dS B_a \tilde{n}^a, \quad (17)$$

$$\tilde{Q}_e^{(n)} = \frac{1}{4\pi} \int_{S_{r \rightarrow \infty}} dS \tilde{E}_j \tilde{n}^j, \quad \tilde{Q}_m^{(n)} = \frac{1}{4\pi} \int_{S_{r \rightarrow \infty}} dS \tilde{B}_j \tilde{n}^j, \quad (18)$$

allied with *electric* and *magnetic* components of the gauge strength fields $F_{\alpha\beta}^{(n)}$ and $\tilde{F}_{\alpha\beta}^{(n)}$, respectively.

Now we shall take into account the axisymmetric initial data, i.e., data that are invariant under the action of $U(1)$ group. On this account axisymmetric feature is encoded in the line element of the form

$$ds^2 = q_{AB} dx^A dx^B + X^2 \left(d\varphi + W_B dx^B \right)^2, \quad (19)$$

where q_{AB} is a two-dimensional metric on the orbit space of Killing vector $\eta_\alpha = (\partial/\partial\varphi)_\alpha$ and moreover the functions X and W_B are independent on φ -coordinate. It turns out that the *strongly axisymmetric* condition input additional mirror symmetry and causes that W_B has to disappear [3]. One can find such coordinate that

$$ds^2 = e^{-2U+2\alpha} (d\rho^2 + dz^2) + \rho^2 e^{-2U} \left(d\varphi + \rho W_\rho d\rho + W_z dz \right)^2, \quad (20)$$

where all the functions are φ -independent. The above choice of the line element leads to finding a harmonic function on the orbit space, i.e., $\nabla_i \nabla_{(q_{AB})}^i \rho = 0$ and specifying conditions at infinity and on the z -axis. Moreover certain conditions on functions U and α in order to obtain regularity of the axisymmetric line element should be imposed [3, 6].

As we shall exploit the *axisymmetric initial data* which make the group of manifold isometries include $U(1)$ -subgroup, the defined quantities should be invariant under the aforementioned group action. Namely, we have that

$$\mathcal{L}_\eta h_{ab} = \mathcal{L}_\eta K_{ij} = \mathcal{L}_\eta E_i^{(n)} = \mathcal{L}_\eta \tilde{E}_i^{(n)} = \mathcal{L}_\eta B_j^{(n)} = \mathcal{L}_\eta \tilde{B}_j^{(n)} = 0, \quad (21)$$

where \mathcal{L}_η is Lie derivative with respect to the Killing vector field η_α . In Ref.[19] it was revealed that in EM-theory the angular momentum in the direction of the rotation axis, of two-dimensional surface $\Sigma \in M$, with a tangent vector η_α and \tilde{n}_i unit outer normal over the coordinate sphere, can be written as

$$J(\Sigma) = \frac{1}{8\pi} \int_\Sigma d\Sigma \left(K_{ij} - K_a^a h_{ij} \right) \tilde{n}^i \eta^j. \quad (22)$$

One should comment that equation (22) describes the *Komar-like* angular momentum connected with a two-dimensional surface with the axial vector η^i , that coincides with the Komar definition of angular momentum when η^i can be expressed in the vicinity of Σ .

But $J(\Sigma)$ is not necessary conserved. The crucial point is that we consider the matter fields like $U(1)$ -gauge fields, dilaton and axion fields, which the standard bulk contribution may be written in terms of Stoke's theorem, using a surface term in a natural way associated with black hole. Hence, one is motivated to define the total angular momentum on a hypersurface Σ , with contributions of gauge fields in the underlying theory, which has this property. Having in mind this idea, we postulate the total angular momentum provided by the following expression:

$$\begin{aligned} \tilde{J}(\Sigma) &= \frac{1}{8\pi} \int_\Sigma d\Sigma \left(K_{ij} - K_a^a h_{ij} \right) \tilde{n}^i \eta^j \\ &+ \frac{1}{4\pi} \sum_{n=1}^N \int_\Sigma d\Sigma A_k^{(n)} \eta^k \tilde{n}_i \tilde{B}^{i(n)}. \end{aligned} \quad (23)$$

The motivation for introducing the second term was mainly to obtain the conservation of the total angular momentum. Moreover, one has that if we set dilaton and axion fields equal to zero and restrict considerations to the only one gauge field, we arrive at the form of the potential in Einstein-Maxwell (EM) theory [19]. On the other hand, the form depending on $\sum_{n=1}^N \tilde{B}^{i(n)}$ (not like in EM-theory on E_j) has its roots in equation of motion for EMAD-gravity.

Namely, in the theory under consideration one has that the divergence of $\sum_{n=1}^N \tilde{B}^{i(n)}$ is equal to zero. Contrary to Maxwell electrodynamics when $\nabla_j E^j = 0$.

Despite of the fact that the potentials $A_j^{(n)}$ are discontinuous on the z -axis, the product $\sum_{n=1}^N A_k^{(n)} \eta^k$ remains well behaved, because of the fact that the Killing vector field η_α vanish on the z -axis.

Now we restrict our attention to the problem of the total angular momentum in the theory under consideration. To proceed further, we shall consider the second term on the right hand-side of relation (12). It yields

$$\begin{aligned} \epsilon_{kij} \tilde{B}^i B^j \eta^k &= \epsilon_{kij} \tilde{B}^i \epsilon^{jab} D_a A_b \eta^k = D_a \left(\epsilon_{kij} \tilde{B}^i \epsilon^{jab} A_b \eta^k \right) \\ &- \epsilon_{kij} D_a \tilde{B}^i \epsilon^{jab} A_b \eta^k - \epsilon_{kij} \tilde{B}^i \epsilon^{jab} A_b D_a \eta^k \\ &= D_a \left(\epsilon_{kij} \tilde{B}^i \epsilon^{jab} A_b \eta^k \right) + A_k \eta^k D_j \tilde{B}^j, \end{aligned} \quad (24)$$

where we have used the invariance properties under the motion of $U(1)$ group. Consequently, let us take into account a domain of the manifold in question, $M_1 \in M$, with boundaries $\partial M_1 = \Sigma_1 \cup \Sigma_2$

$$\int_{M_1} dV \epsilon_{kij} \tilde{B}^i B^j \eta^k = \int_{M_1} dV A_k \eta^k D_j \tilde{B}^j - \int_{\partial M_1} d\Sigma A_k \eta^k \tilde{B}^j \tilde{n}_j. \quad (25)$$

In derivation of the above equation one has to take into account that the Killing vector field η_j is perpendicular to \tilde{n}^j vector. Due to the fact that $D_a \tilde{B}^a = 0$, one arrives at

$$\begin{aligned} \int_{M_1} dV P_a \eta^a &= \frac{1}{8\pi} \int_{\partial M_1} d\Sigma \left(K_{ij} - K_a^a h_{ij} \right) \eta^i \tilde{n}^j + \frac{1}{4\pi} \sum_{n=1}^N \int_{\partial M_1} d\Sigma A_k^{(n)} \eta^k \tilde{n}_i \tilde{B}^{i(n)} \\ &= \tilde{J}(\Sigma_2) - \tilde{J}(\Sigma_1). \end{aligned} \quad (26)$$

If the left-hand side of the above relation is equal to zero the total angular momentum is conserved. Moreover, it can be revealed, using the definition of $\tilde{J}(\Sigma)$, that this quantity is invariant with respect to the gauge transformation $A_i \rightarrow A_i + D_i \theta$. Of course one ought to assume that θ disappear near infinity and equation (21) is fulfilled.

Next we consider the behaviour of the total angular momentum near infinity. It suffices to examine the expression given by

$$\int_{S_{r \rightarrow \infty}} d\Sigma \tilde{B}_i^{(n)} \tilde{n}^i A_k^{(n)} \eta^k, \quad (27)$$

for each of the gauge field in the theory in question. We assume that $A_k^{(n)} \sim \mathcal{O}(1/r)$, $\tilde{A}_k^{(n)} \sim \mathcal{O}(1/r)$ and for the Killing vector field one has that $|\eta| \sim |x \partial_y - y \partial_x| = \mathcal{O}(\rho)$. On the other hand, for the *magnetic* and *electric* one suppose that they are proportional to $\partial_r/r^2 + \mathcal{O}(1/r^3)$. In Ref.[19] the typical construction avoiding the difficulty of removing Dirac string bounded with each asymptotical point i_k was performed. One removes from the manifold in question the portion of the z -axis below or above the adequate asymptotical point. The aforementioned method enables one to obtain $U(1)$ invariant potential for each of the gauge field $A_k^{(n)}$ in the form as

$$A_i^{(n)} = \frac{1}{2k} \sum_{k=1}^M \left(A_{+i}^{(n)} + A_{-i}^{(n)} \right), \quad (28)$$

on $R^3 \setminus \{z\text{-axis}\}$. Having in mind the asymptotical behaviours described above one can show that

$$\lim_{r \rightarrow \infty} \frac{1}{r^2} \frac{1}{2k} \sum_{k=1}^M \sum_{n=1}^N \left(A_{+i}^{(n)} + A_{-i}^{(n)} \right) \eta^i = 0. \quad (29)$$

Just the total angular momentum \tilde{J} tends at infinity to $J(\Sigma)$. Summing it all up, one can formulate the statement Theorem:

Let $(M, h_{ij}, K_{ab}, E_i, \tilde{E}_i, B_a, \tilde{B}_a, \lambda)$ be initial axisymmetric data of the quantities defined above. If $P_k \eta^k = 0$, then \tilde{J} is concerned, i.e., that for two $U(1)$ invariant hypersurfaces Σ_1 and Σ_2 and bounded domain one has that

$$\tilde{J}(\Sigma_1) = \tilde{J}(\Sigma_2). \quad (30)$$

Moreover, \tilde{J} is invariant under the gauge transformations vanishing in the nearby of asymptotic regions, following, that

$$\tilde{J}(S_\infty) = J. \quad (31)$$

By analogy with Einstein-Maxwell theory, we would like to pay some attention to the problem of the so-called twist potential. It yields

$$d\lambda = \epsilon_{abc} \left(\pi^{bk} - 2 \sum_{n=1}^N \theta^{(n)bk} \right) \eta^c \eta_k dx^a, \quad (32)$$

where

$$\pi_{ab} = K_{ab} - K_c{}^c h_{ab}, \quad \theta_{ab}^{(n)} = \epsilon_{imb} \tilde{B}^{(n)i} \epsilon_a{}^{lm} A_l^{(n)} \eta^b. \quad (33)$$

It will be interesting to elaborate conditions for which the twist potential exists. Namely, we calculate $(d\lambda)_{ij}$. After using equation (12) and properties of Killing vector fields η_α it can be found that the following is fulfilled:

$$\begin{aligned} (d\lambda)_{ij} &= D^a \left(\pi_{ab} \eta^b - 2 \sum_{n=1}^N \theta_{ab}^{(n)} \eta^b \right) \epsilon_{ijl} \eta^l \\ &= \left(8\pi P_j \eta^j + 2 \sum_{n=1}^N D_i \tilde{B}^i A_b \eta^b \right) \epsilon_{ijl} \eta^l. \end{aligned} \quad (34)$$

From the above relation one has that if $P_a \eta^a = 0$ the twist potential form is closed, i.e., $(d\lambda)_{ij} = 0$. Moreover, as we assumed previously the manifold in question is simply connected and all these facts imply that the twist potential exist.

In [3, 6] it was shown that the ADM mass m can be written in the form as

$$m = \frac{1}{16\pi} \int dx^3 \left[{}^{(3)}R + \frac{1}{2} \rho^2 e^{-4\alpha+2U} \left(\rho W_{\rho,z} - W_{z,\rho} \right)^2 \right] e^{2\alpha-2U} + \frac{1}{8\pi} \int dx^3 (DU)^2. \quad (35)$$

Because of the fact that we consider a simply connected manifold, one enables to justify the existence of the potentials for each of the gauge field. This implies the following relations:

$$\nabla_\alpha \zeta^{(n)} = F_{\alpha\mu}^{(n)} \eta^\alpha, \quad \nabla_\alpha \psi^{(n)} = *F_{\alpha\mu}^{(n)} \eta^\alpha, \quad (36)$$

$$\nabla_\alpha \tilde{\zeta}^{(n)} = \tilde{F}_{\alpha\mu}^{(n)} \eta^\alpha, \quad \nabla_\alpha \tilde{\psi}^{(n)} = *\tilde{F}_{\alpha\mu}^{(n)} \eta^\alpha, \quad (37)$$

In the orthonormal basis one has that

$$\partial_\alpha \Phi = \sqrt{g_{\varphi\varphi}} \Gamma_{3\alpha}^{(n)}, \quad (38)$$

where $\Phi = (\zeta^{(n)}, \tilde{\zeta}^{(n)}, \psi^{(n)}, \tilde{\psi}^{(n)})$ and $\Gamma_{3\alpha}^{(n)} = (F_{\alpha\mu}^{(n)}, *F_{\alpha\mu}^{(n)}, \tilde{F}_{\alpha\mu}^{(n)}, \tilde{F}_{\alpha\mu}^{(n)})$. In Ref.[5] it was found that the potential was bounded with the extrinsic curvature tensor K_{ij} by the relation of the form

$$\lambda = 2 \epsilon_{ijk} K^j{}_l \eta^k \eta^l dx^i. \quad (39)$$

On the other hand, one can find that the following equality is valid:

$$e^{2\alpha-2U} |K|_h^2 \geq 2 e^{2\alpha-2U} (K_{13}^2 + K_{23}^2) = \frac{e^{4U}}{2\rho^4} |\lambda|_h^2. \quad (40)$$

We also assume that the initial data set in maximal, i.e., $K_j{}^j = 0$. Then, we insert equation (40) into relation (35). The outcome is provided by

$$m \geq \frac{1}{16\pi} \int dx^3 \left[{}^{(3)}R e^{2\alpha-2U} + 2 (DU)^2 \right] \quad (41)$$

$$\geq \frac{1}{16\pi} \int dx^3 \left[(DU)^2 + \frac{e^{4U}}{2\rho^4} |\lambda|_h^2 + 2 \frac{e^{2U}}{\rho^2} \sum_{n=1}^N \left(D\psi^{(n)} D\tilde{\zeta}^{(n)} - D\zeta^{(n)} D\tilde{\psi}^{(n)} \right) \right]. \quad (42)$$

It happened that, the last term in the above inequality can be rearrange in the way given by

$$B_i^{(n)} \tilde{E}^{(n)i} - E_j^{(n)} \tilde{B}^{(n)j} = e^{-2\phi} \frac{e^{2U}}{\rho^2} \left[(D\zeta^{(n)})^2 + (D\psi^{(n)})^2 \right], \quad (43)$$

where we have taken into account the definition of $\tilde{F}_{ab}^{(n)}$. Further one can define *dilaton-electric* and *dilaton-magnetic* charges and the adequate potentials for each of the gauge fields under considerations. Namely, they are provided by the relations of the forms as

$$\nabla_\alpha \hat{\psi}^{(n)} = e^{-\phi} * F_{\beta\alpha}^{(n)} \eta^\beta, \quad Q_{d-e}^{(n)} = -\frac{1}{4\pi} \int dS_{\alpha\beta} e^{-\phi} * F^{\alpha\beta(n)} = \frac{\hat{\psi}_2^{(n)} - \hat{\psi}_1^{(n)}}{2}, \quad (44)$$

$$\nabla_\alpha \hat{\zeta}^{(n)} = e^{-\phi} F_{\beta\alpha}^{(n)} \eta^\beta, \quad Q_{d-m}^{(n)} = -\frac{1}{4\pi} \int dS_{\alpha\beta} e^{-\phi} F^{\alpha\beta(n)} = \frac{\hat{\zeta}_1^{(n)} - \hat{\zeta}_2^{(n)}}{2}. \quad (45)$$

By virtue of the above definitions we can invoke all the procedure elaborated in Refs.[6]-[9], to find an inequality binding the black hole ADM mass with other quantities characterizing black hole in EMAD-gravity. Namely, one defines action

$$I = \int dx^3 \left[(DU)^2 + \frac{e^{4U}}{2\rho^4} |\lambda|_h^2 + 2 \frac{e^{2U}}{\rho^2} \sum_{n=1}^N \left((D\hat{\zeta}^{(n)})^2 + (D\hat{\psi}^{(n)})^2 \right) \right]. \quad (46)$$

Then we use harmonic map associated with the extreme Kerr-Sen solution $I(\tilde{\mu}, \tilde{\lambda}, \tilde{\zeta}, \tilde{\psi})$. One would like to show that $I(\tilde{\mu}, \tilde{\lambda}, \tilde{\zeta}, \tilde{\psi}) \geq I(\mu, \lambda, \hat{\zeta}, \hat{\psi})$. It can be shown by the methods developed in Refs. [6]-[9], so we refer readers to the original works for particulars. Consequently, we can conclude that

Theorem:

Let $(M, h_{ij}, K_{ij}, \hat{\zeta}, \hat{\psi}, \lambda)$ be a smooth three-dimensional maximal time symmetric data set on simply connected manifold which is invariant under the action of $U(1)$ -group, with two asymptotically flat regions. Suppose further that there is no electromagnetic matter fields and the dominant energy condition is assured. Then the ADM mass m , angular momentum and global *dilaton-electric* and *dilaton-magnetic* charges one obtains the inequality

$$m \geq \sqrt{\frac{|\vec{J}|^2}{m^2} + \sum_{n=1}^N \left(Q_{d-e}^{(n)2} + Q_{d-m}^{(n)2} \right)}. \quad (47)$$

By the direct calculations one can readily verify that the above inequality can be rewritten in the analogous form

$$m^2 \geq \frac{Q_{(N)}^2 + \sqrt{Q_{(N)}^4 + 4 |\vec{J}|^2}}{2}, \quad (48)$$

where for the brevity of notation we set $Q_{(N)}^2$ equal to

$$Q_{(N)}^2 = \sum_{n=1}^N \left(Q_{d-e}^{(n)2} + Q_{d-m}^{(n)2} \right). \quad (49)$$

IV. AREA INEQUALITIES

In this section we comment on the inequality connecting the area, angular momentum and charges for dynamical black holes in EMAD-gravity. To commence with, one considers a closed orientable two-dimensional spacelike surface \mathcal{S} smoothly embedded in the manifold in question. Its intrinsic geometry is characterized by induced metric q_{ab} with connection ${}^{(2)}D_a$, Ricci scalar ${}^{(2)}R$, volume element ϵ_{ab} and area measure $d\mathcal{S}$. As far as the extrinsic geometry is concerned one introduces the normal outgoing and ingoing null vectors l^i and k^j normalized as $l^i k_i = -1$. Therefore the extrinsic geometry is characterized by the expansion $\theta^{(l)}$, the shear $\sigma_{ij}^{(l)}$ and the normal fundamental form $\Omega_j^{(l)}$ bounded with the outgoing normal null vector l^a . They are provided by the relations as follows:

$$\begin{aligned} \theta^{(l)} &= q^{ab} \nabla_a l_b, & \sigma_{ij}^{(l)} &= q^c{}_i q^d{}_j \nabla_c l_d - \frac{1}{2} \theta^{(l)} q_{ij}, \\ \Omega_j^{(l)} &= -k^m q^r{}_j \nabla_m l_r. \end{aligned} \quad (50)$$

Moreover we require that the surface \mathcal{S} is the marginally outer trapped surface, i.e., $\theta^{(l)} = 0$, as well as we demand that the hypersurface in question is stable. It means that there is an outgoing vector $X^a = \lambda_1 l_a - \lambda_2 k_a$, with $\lambda_1 \geq 0$

and $\lambda_2 > 0$ satisfying the condition of the form as $\delta_X \theta^{(l)} \geq 0$. The operator δ_X is the variation operator on surface \mathcal{S} along the vector X^a [14]. Additionally the surface should be axisymmetric with the Killing vector field η_a and the following relations should be given

$$\mathcal{L}_\eta l^j = \mathcal{L}_\eta k^j = \mathcal{L}_\eta \Omega_j^{(l)} = \mathcal{L}_\eta F^{(n)} = \mathcal{L}_\eta \lambda = 0, \quad (51)$$

where $F^{(n)}$ denotes the projection of the adequate strength of n -th gauge field.

In Ref.[14] it was revealed that for a closed marginally trapped surface \mathcal{S} satisfying the stably outermost condition for vector X^a and for every axisymmetric function α , the following inequality implied

$$\begin{aligned} \int_{\mathcal{S}} dS \left({}^{(2)}D_a \alpha {}^{(2)}D^a \alpha + \frac{1}{2} \alpha^2 {}^{(2)}R \right) \geq \int_{\mathcal{S}} dS \left(\alpha^2 \Omega_j^{(\eta)} \Omega^{(\eta)j} + \alpha \beta \sigma_{ij}^{(l)} \sigma^{(l)ij} \right. \\ \left. + G_{ab} \alpha l^a (\alpha k^b + \beta l^b) \right), \end{aligned} \quad (52)$$

where $\beta = \alpha \lambda_1 / \lambda_2$. In the case of EMAD-gravity the right-hand side of the above inequality is equal to relation

$$\begin{aligned} \int_{\mathcal{S}} dS \left[\alpha^2 \Omega_j^{(\eta)} \Omega^{(\eta)j} + \alpha \beta \sigma_{ij}^{(l)} \sigma^{(l)ij} + 2 \alpha \beta (l^j \nabla_j \phi)^2 + \frac{1}{2} e^{4\phi} \alpha^2 (l^j \nabla_j a)^2 \right. \\ \left. + 2 \alpha \beta \sum_{n=1}^N (F_{ak}^{(n)} l^a) (F_j^{(n)k} l^j) + T_{ij}(\text{matter}) \alpha l^i (\alpha k^j + \beta l^j) \right. \\ \left. + \alpha^2 \left(\Omega_j^{(\eta)} \Omega^{(\eta)j} + {}^{(2)}D_i \phi {}^{(2)}D^i \phi + \frac{1}{4} e^{4\phi} {}^{(2)}D_k a {}^{(2)}D^k a + e^{-2\phi} \sum_{n=1}^N (\mathcal{E}^2 + \mathcal{B}^2) \right) \right], \end{aligned} \quad (53)$$

where $\mathcal{E} = F_{ab} l^a k^b$ and $\mathcal{B} = *F_{ij} l^i k^j$. Because of the fact that we assume the dominant energy condition for matter fields, i.e., $T_{ij}(\text{matter}) \alpha l^i (\alpha k^j + \beta l^j) \geq 0$ as well as null energy condition for $U(1)$ -gauge fields $2 \alpha \beta \sum_{n=1}^N (F_{ak}^{(n)} l^a) (F_j^{(n)k} l^j) \geq 0$, one obtains all positive terms on the right-hand side of equation (53). Abandoning the non-negative terms we arrive at the following relation:

$$\begin{aligned} \int_{\mathcal{S}} dS \left({}^{(2)}D_a \alpha {}^{(2)}D^a \alpha + \frac{1}{2} \alpha^2 {}^{(2)}R \right) \geq \\ \int_{\mathcal{S}} dS \alpha^2 \left[\Omega_j^{(\eta)} \Omega^{(\eta)j} + {}^{(2)}D_i \phi {}^{(2)}D^i \phi + \frac{1}{4} e^{4\phi} {}^{(2)}D_k a {}^{(2)}D^k a + e^{-2\phi} \sum_{n=1}^N (\mathcal{E}^2 + \mathcal{B}^2) \right]. \end{aligned} \quad (54)$$

To have a closer insight in the inequality we introduce the following axisymmetric line element on the two-dimensional surface \mathcal{S}

$$ds^2 = q_{ab} dx^a dx^b = e^\sigma \left(e^{2q} d\theta^2 + \sin^2 \theta d\varphi^2 \right), \quad (55)$$

where $\sigma + q = \text{const} = c$. Now it should be recalled [14] that the fundamental form $\Omega_a^{(l)}$ can be festered by means of the Hodge decomposition, i.e., $\Omega_a^{(l)} = \epsilon_{ab} D^b \omega + D_a \tilde{\lambda}$. Because of the fact that $\Omega_a^{(l)}$ is axisymmetric one can readily verify that $\Omega_a^{(\eta)} = \frac{1}{2\eta} \epsilon_{ab} D^b \omega$. It was revealed in the preceding section that the total angular momentum consist of the gravitational part and the element contributed to the gauge fields in the underlying theory. The gravitational branch of the angular momentum is given by

$$J = \frac{1}{8\pi} \int_{\mathcal{S}} dS \Omega_a^{(l)} \eta^a = \frac{\omega(\pi) - \omega(0)}{8}. \quad (56)$$

In order to describe the other part of the potential it is useful to introduce another potential [32] of the form

$$d\chi = \sum_{n=1}^N \left(2 \eta d\omega - 2v^{(n)} dk^{(n)} + 2 k^{(n)} dv^{(n)} \right). \quad (57)$$

Then, by a direct calculation it can be revealed that $d\mathcal{S} = e^c dS_0$, where $dS_0 = \sin\theta d\theta d\varphi$. Choosing $\alpha = e^{c-\sigma/2}$ one achieves at

$$\begin{aligned} 2(c+1) \geq & \frac{1}{2\pi} \int_S dS_0 \left[\sigma + \frac{1}{4} D_m \sigma D^M \sigma + D_a \phi D^a \phi + \frac{1}{4} e^{4\phi} D_i a D^i a \right. \\ & + \frac{1}{4\eta^2} \left| \sum_{n=1}^N \left(D_j \chi + 2v^{(n)} D_j k^{(n)} - 2 k^{(n)} D_j v^{(n)} \right) \right|^2 \\ & \left. + \frac{1}{\eta} \sum_{n=1}^N D_a \hat{\psi}^{(n)} D^a \hat{\psi}^{(n)} + \frac{1}{\eta} \sum_{n=1}^N D_b \hat{\zeta}^{(n)} D^b \hat{\zeta}^{(n)} \right], \end{aligned} \quad (58)$$

where $\eta = q_{ij} \eta^i \eta^j$. Having in mind that $A = 4\pi e^c$ one can reach the inequality

$$A \geq 4\pi e^{\frac{\mathcal{M}-2}{2}}, \quad (59)$$

where the functional \mathcal{M} is defined as the right-hand side of the equation (58).

In order to the inequality among area, angular momentum and charges is to utilize the connection between the functional \mathcal{M} and a harmonic energy for maps from the sphere into the complex hyperbolic space. The key point in the proof is to show that the extreme Kerr-Sen sphere, i.e., the set fulfilling the Lagrange equations for the functional \mathcal{M} [12].

One should also take into account the results of Ref.[33], which state that if the domain for the map is compact, connected, with non-void boundary and the target manifold has negative sectional curvature, then the minimizer of the harmonic energy subject to the Dirichlet boundary conditions, exists. Just, on account of this result one can conclude that harmonic maps are minimizers of the harmonic energy for the given Dirichlet boundary conditions. As in Ref.[16] we relate the functional \mathcal{M} to the standard harmonic energy \mathcal{M}_D from a subset $D \subset S^2 \setminus \{\theta = 0, \pi\}$ to the complex hyperbolic space with the strictly positive line element provided by

$$ds_H^2 = \frac{d\eta^2}{\eta} + \frac{1}{\eta^2} \left[\sum_{n=1}^N (d\chi + 2v^{(n)} dk^{(n)} - 2 k^{(n)} dv^{(n)}) \right]^2 + \frac{1}{\eta} \sum_{n=1}^N \left[(d\hat{\zeta}^{(n)})^2 + (d\hat{\psi}^{(n)})^2 \right], \quad (60)$$

while \mathcal{M}_D implies

$$\mathcal{M}_D = \mathcal{M} + \int_S dS \ln \sin \theta + \int_{\partial S} dl (\sigma + \ln \sin \theta) \frac{\partial \ln \sin \theta}{\partial n}. \quad (61)$$

We have set n to be unit normal to the boundary to \mathcal{S} surface, while dl is the measure element of the boundary $\partial\mathcal{S}$. It should be noticed that both functionals have the same forms of the Lagrange equations because of the fact that the difference between them is equal to a constant plus a boundary term. The proof goes like in Ref.[16] so we refer the reader for the mathematical details to the article. For the convenience of the reader we sketch the main steps of it. Namely, we divide the underlying sphere into three regions

$$\Omega_I = \{\sin \theta \leq e^{-(\ln \epsilon)^2}\}, \quad \Omega_{II} = \{e^{-(\ln \epsilon)^2} \leq \sin \theta \leq \epsilon\}, \quad \Omega_{III} = \{\epsilon \leq \sin \theta\}. \quad (62)$$

Firstly, we interpolate the potentials between extreme Kerr-Sen solution in Ω_I and a general solution in Ω_{III} region. It leads to the Dirichlet problem in $\Omega_{IV} = \Omega_{II} \cup \Omega_{III}$ which yields that the mass functional of Kerr-Sen extreme solution is less than or equal to the mass functional for the auxiliary interpolating map on the whole sphere. In the last step one has in mind the limit as Ω_{III} converges to the sphere and reveals that the mass functional in question for the auxiliary maps converges to the mass functional for the original sets. All these mathematical machinery leads to the inequality

$$e^{\mathcal{M}-2} \geq 4 J^2 + Q_{(N)}^4. \quad (63)$$

One can make use of the above inequality. In the case when we consider two asymptotically flat ends there exist an asymptotic stable (i.e., the second variation of the area is nonnegative) minimal surface $\Sigma_{min} \in \mathcal{M}$ which separates the aforementioned asymptotically flat ends. As was remarked in [19] Σ_{min} minimizes area among all the considered two-surfaces $A(\Sigma_{min}) = A_{min}$, where A_{min} is the least area pone requires to enclose the ends. Just having in mind conclusions presented in Ref.[19] and our inequality one obtains

$$A_{min} \geq 4\pi \sqrt{4 J^2(\Sigma) + Q_{(N)}^4(\Sigma)}, \quad (64)$$

where in the above inequality Σ stands either for Σ_{min} or for Σ_0 . The equality is satisfied when $\Sigma = \Sigma_0$ and this case is responsible for the Kerr-Sen extreme sphere. Using the relations (47) and (64), one can conclude

Theorem:

Assume that one has axially symmetric, maximal and simply connected initial data set with two asymptotically flat ends. Suppose moreover that we consider non-electromagnetic matter field in EMAD-gravity. Let us demand that the dominant energy condition and $P_k \eta^k = 0$ is fulfilled, where η_k is the axially symmetric Killing vector field. Then, the following inequality is provided:

$$\frac{A_{min}}{8\pi} \geq m^2 - \frac{Q_{(N)}^2}{2} - \sqrt{\left(m^2 - \frac{Q_{(N)}^2}{2}\right)^2 - \frac{Q_{(N)}^4}{4} - J^2}, \quad (65)$$

where A_{min} is the minimum area to enclose the asymptotically flat ends. The Kerr-Sen extremal spacetime is subject to the equality.

V. CONCLUSIONS

In our paper we have considered the EMAD-gravity theory being the low-energy limit of the heterotic string theory with arbitrary number of $U(1)$ -gauge fields. Matter fields were assumed to be non-electromagnetic and satisfying the dominant energy condition. We define the general form of the total angular momentum as well as the twist potential in the theory under consideration. Considering a smooth three-dimensional time symmetric data set on simply connected manifold invariant under the action of $U(1)$ group, with two asymptotically flat ends, we arrive at the inequality binding the ADM mass angular momentum and global *dilaton-electric* and *dilaton-magnetic* charges. Then, we examine a closed orientable two-dimensional spacelike surface which is smoothly embedded in spacetime of EMAD-gravity. Then it was shown that the ADM mass is subject to inequality expressed in terms of the area angular momentum and charges of black holes in EMAD-gravity.

Considering axially symmetric maximal and simply connected data set with two-asymptotically flat ends and demanding the dominant energy condition and relation $P^a \eta_a = 0$, we achieved at the inequality expressing the area in terms of the ADM mass, angular momentum and charges in the underlying theory. The inequality was saturated if the initial data emerge from the extremal Kerr-Sen black hole.

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